

Modified wave operators without loss of regularity for some long range Hartree equations. II

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Abstract

We continue the study of the theory of scattering for some long range Hartree equations with potential $|x|^{-\gamma}$, performed in a previous paper, denoted as I, in the range $1/2 < \gamma < 1$. Here we extend the results to the range $1/3 < \gamma < 1/2$. More precisely we study the local Cauchy problem with infinite initial time, which is the main step in the construction of the modified wave operators. We solve that problem without loss of regularity between the asymptotic state and the solution, as in I, but in contrast to I, we are no longer able to cover the entire subcriticality range of regularity of the solutions. The method is an extension of that of I, using a better approximate asymptotic form of the solutions obtained as the next step of a natural procedure of successive approximations.

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1 Introduction

This paper is the continuation of a previous paper [5], hereafter referred to as I, where we studied the theory of scattering and more precisely the proof of existence of modified wave operators for the long range Hartree type equation

$$i\partial_t u = -(1/2)\Delta u + g(u)u . \quad (1.1)$$

Here u is a complex valued function defined in space time \mathbb{R}^{n+1} with $n \geq 2$, Δ is the Laplace operator in \mathbb{R}^n and

$$g(u) = \kappa |x|^{-\gamma} \star |u|^2 \quad (1.2)$$

where $\kappa \in \mathbb{R}$, $0 < \gamma \leq 1$ and \star denotes the convolution in \mathbb{R}^n .

The main step of that existence proof consists in solving the local Cauchy problem with infinite initial time for (1.1), namely to construct solutions thereof with prescribed asymptotic behaviour as $t \rightarrow \pm\infty$. We refer to the introduction of I and to [2] [3] [6] [7] for general background. In the long range situation $\gamma \leq 1$ that we consider, the asymptotic behaviour of u includes a phase which diverges at infinity in time, and is parametrized by an asymptotic state which plays the role of initial data at infinity. In [2], we solved the previous local Cauchy problem at infinity in the range $1/2 < \gamma < 1$ (the easier borderline case $\gamma = 1$ can be treated by the same method), but the treatment in [2] involved a loss of regularity between the asymptotic state and the solution and failed to cover the entire natural subcritical range of regularity for the asymptotic state. These two defects were remedied in [6] and [7] in the cases $\gamma = 1$ and $1/2 < \gamma < 1$ respectively. The main results of [6] [7] were then recovered in I by what we regard as a simpler method combining ingredients of [2] and [6]. On the other hand, the problem was solved in [3] by an extension of the method of [2] for $\gamma \leq 1/2$, actually for any $\gamma \leq 1$, again with a loss of regularity between the asymptotic state and the solution. That loss increases as γ decreases through inverse integer values. Now it turns out that the simple method of I can be extended below $\gamma = 1/2$ to solve the problem without loss of regularity. For $\gamma < 1/2$ however, it no longer allows to cover the whole subcritical range, and stronger regularity of the asymptotic state is needed. Furthermore the treatment, although still elementary, becomes increasingly cumbersome as γ decreases through inverse integer values. The present paper is devoted to the application of that method to the next accessible range, namely for $1/3 < \gamma < 1/2$, as an illustration of that possibility.

The easier borderline case $\gamma = 1/2$ can be treated by the same method, but requires slightly different formulas.

We now introduce the relevant parametrization of u needed to study the Cauchy problem at infinite time, restricting our attention to positive time. The unitary group

$$U(t) = \exp(i(t/2)\Delta) \quad (1.3)$$

which solves the free Schrödinger equation can be written as

$$U(t) = M(t) D(t) F M(t) \quad (1.4)$$

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp(ix^2/2t) , \quad (1.5)$$

F is the Fourier transform and $D(t)$ is the dilation operator

$$D(t) = (it)^{-n/2} D_0(t) \quad (1.6)$$

where

$$(D_0(t)f)(x) = f(x/t) . \quad (1.7)$$

For any function w of space time, we define

$$\tilde{w}(t) = U(-t) w(t) \quad (1.8)$$

and we define the pseudoconformal inverse w_c of w by

$$w(t) = M(t) D(t) \overline{w_c(1/t)} \quad (1.9)$$

or equivalently

$$\tilde{w}(t) = \overline{F\tilde{w}_c(1/t)} \quad (1.10)$$

which shows that the pseudoconformal inversion is involutive.

The Cauchy problem at infinite initial time for u is then equivalent to the Cauchy problem at initial time zero for its pseudoconformal inverse u_c . The equation (1.1) is replaced by

$$i\partial_t u_c = -(1/2)\Delta u_c + t^{\gamma-2} g(u_c)u_c . \quad (1.11)$$

We now parametrize u_c in terms of an amplitude v and a phase φ according to

$$u_c(t) = \exp(-i\varphi(t))v(t) \quad (1.12)$$

so that

$$\begin{aligned} u(t) &= M(t) D(t) \exp(i\varphi(1/t)) \bar{v}(1/t) \\ &= D(t) \exp(i\varphi(1/t)) D^{-1}(t) M(t) D(t) \bar{v}(1/t) \end{aligned}$$

or equivalently

$$u(t) = \exp(i(D_0(t)\varphi(1/t))) v_c(t) . \quad (1.13)$$

The original equation then becomes the following equation for v

$$i\partial_t v = -(1/2)\Delta_s v + \left(t^{\gamma-2}g(v) - \partial_t\varphi\right)v \quad (1.14)$$

where $s = \nabla\varphi$ and

$$\Delta_s = (\nabla - i s)^2 = \Delta - 2i s \cdot \nabla - i(\nabla \cdot s) - |s|^2 . \quad (1.15)$$

We want to choose φ so as to cancel the divergence at $t = 0$ of the last term in (1.14), but that cancellation is needed only at large distances, namely for low momentum. We therefore introduce a momentum cut-off as follows. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^+)$, $0 \leq \chi \leq 1$, $\chi(\ell) = 1$ for $\ell \leq 1$, $\chi(\ell) = 0$ for $\ell \geq 2$. We define

$$\chi_L = \chi(\omega t^{1/2}) \quad , \quad \chi_S = 1 - \chi_L \quad (1.16)$$

with $\omega = (-\Delta)^{1/2}$, and correspondingly

$$g_L(v) = \chi_L g(v) \quad , \quad g_S(v) = \chi_S g(v) . \quad (1.17)$$

We want to solve (1.14) with v continuous at $t = 0$ with $v(0) = v_0$ for a given v_0 . In I we chose φ such that

$$\partial_t\varphi - t^{\gamma-2} g_L(v_0) \approx 0 , \quad (1.18)$$

a choice which was sufficient for $\gamma > 1/2$. That choice however is not sufficient for $\gamma \leq 1/2$ since then the terms coming from $|s|^2$ and from $g_L(v) - g_L(v_0)$ in (1.14) both fail to be integrable at $t = 0$. We must therefore choose a better asymptotic form v_a for v , still with $v_a(0) = v_0$. We rewrite (1.14) as

$$i\partial_t v = L(v)v \quad (1.19)$$

with

$$\begin{aligned} L(v) &= -(1/2)\Delta + i s \cdot \nabla + (i/2)(\nabla \cdot s) + t^{\gamma-2}g_S(v) \\ &+ t^{\gamma-2}(g_L(v) - g_L(v_a)) + (1/2)|s|^2 + t^{\gamma-2}g_L(v_a) - \partial_t\varphi . \end{aligned} \quad (1.20)$$

If the asymptotic v_a is sufficiently accurate, we may expect that the term with $g_L(v) - g_L(v_a)$ will be integrable at $t = 0$ and we may try to cancel the remaining divergences by choosing φ according to

$$\partial_t \varphi = t^{\gamma-2} g_L(v_a) + (1/2)|s|^2, \quad (1.21)$$

with initial condition $\varphi(1) = 0$, since the RHS of (1.21) fails to be integrable at $t = 0$. In order to control the term with $g_L(v) - g_L(v_a)$, as in I and following [6], we use the facts that $g(v)$ depends only on $|v|^2$ and that, if v satisfies a linear Schrödinger equation

$$i\partial_t v = -(1/2)\Delta_s v + V v \quad (1.22)$$

for some real potential V , then v satisfies the local conservation law

$$\partial_t |v|^2 = -\operatorname{Im} \bar{v} \Delta v + \nabla \cdot s |v|^2. \quad (1.23)$$

If we impose that v_a satisfies the transport equation

$$\partial_t v_a = s \cdot \nabla v_a + (1/2)(\nabla \cdot s) v_a, \quad (1.24)$$

then we obtain

$$\partial_t (|v|^2 - |v_a|^2) = -\operatorname{Im} \bar{v} \Delta v + \nabla \cdot s (|v|^2 - |v_a|^2) \quad (1.25)$$

which provides a good starting point to estimate $g_L(v) - g_L(v_a)$. (One could also impose the Schrödinger equation

$$i\partial_t v_a = -(1/2)\Delta_s v_a$$

but that would introduce unnecessary complications without improving the crucial estimates).

We are therefore led to choose (φ, v_a) by solving the system (1.21) (1.24) with initial conditions $\varphi(1) = 0$, $v_a(0) = v_0$, but this is a nonlinear system which is hardly simpler than the original equation, and we seem to have gained nothing so far. However (φ, v_a) are only asymptotic quantities, and it suffices to solve that system approximately by iteration. We therefore define successive approximate solutions (φ_m, v_m) of (1.21) (1.24) by

$$\begin{cases} \partial_t \varphi_m = t^{\gamma-2} g_L(v_m) + (1/2)|s_{m-1}|^2 \\ \partial_t v_m = s_{m-1} \cdot \nabla v_m + (1/2)(\nabla \cdot s_{m-1}) v_m \end{cases} \quad (1.26)$$

with $\varphi_m(1) = 0$, $v_m(0) = v_0$. The system (1.26) determines v_m by a linear transport equation with a smooth vector field s_{m-1} and then φ_m by integration over time. The choice made in I was essentially the case $m = 0$ (with $\varphi_{-1} \equiv 0$), namely

$$\begin{cases} \partial_t \varphi_0 = t^{\gamma-2} g_L(v_0) \\ \partial_t v_0 = 0 \end{cases} \quad (1.27)$$

where by a slight abuse of notation we denote by v_0 both the initial value $v_m(0)$ and the constant function of time equal to v_0 . That choice was adequate for $\gamma > 1/2$. In the present paper, we use the next approximation $m = 1$, namely we take $(\varphi, v_a) = (\varphi_1, v_1)$ so that

$$\begin{cases} \partial_t \varphi = t^{\gamma-2} g_L(v_a) + (1/2) |s_0|^2 \\ \partial_t v_a = s_0 \cdot \nabla v_a + (1/2) (\nabla \cdot s_0) v_a \end{cases} \quad (1.28)$$

$$(1.29)$$

with $s_0 = \nabla \varphi_0$ defined by (1.27), with $\varphi_0(1) = \varphi(1) = 0$ and $v_a(0) = v_0$. That choice turns out to be sufficient to cover the range $1/3 < \gamma \leq 1/2$. With that choice, the basic equation to be solved is (1.19), where now

$$\begin{aligned} L(v) = & -(1/2) \Delta + i s \cdot \nabla + (i/2) (\nabla \cdot s) + t^{\gamma-2} g_S(v) \\ & + t^{\gamma-2} (g_L(v) - g_L(v_a)) + (1/2) (|s|^2 - |s_0|^2) \end{aligned} \quad (1.30)$$

and

$$s = s_0 + s_b + s_c, \quad (1.31)$$

$$s_0 = -\nabla \int_t^1 dt' t'^{\gamma-2} g_L(v_0), \quad (1.32)$$

$$s_b = -(1/2) \nabla \int_t^1 dt' |s_0(t')|^2, \quad (1.33)$$

$$s_c = -\nabla \int_t^1 dt' t'^{\gamma-2} (g_L(v_a) - g_L(v_0)) (t'). \quad (1.34)$$

More generally, one expects the approximation (φ_m, v_m) to be sufficient to cover the range $1/(m+2) < \gamma \leq 1/(m+1)$.

In the present paper, we treat the problem in the range $1/3 < \gamma < 1/2$ with the previous choice of (φ, v_a) . (The simpler case $\gamma = 1/2$ can be treated with the same choice, but requires slightly different formulas). As mentioned above, we solve the local Cauchy problem at infinity in time for u (at time zero for v) without any regularity loss between the asymptotic state v_0 and the solution v , but in contrast

to I, we are unable to cover the entire subcritical range for v , and stronger than subcritical regularity is required for v_0 as soon as $\gamma < 1/2$.

In addition to (1.19), we shall also need the partly linearized equation for v'

$$i\partial_t v' = L(v)v' \quad (1.35)$$

with $L(v)$ again defined by (1.30).

The method consists in first solving the Cauchy problem with initial time zero for the linearized equation (1.35). One then shows that the map $v \rightarrow v'$ thereby defined is a contraction in a suitable space in a sufficiently small time interval. This solves the Cauchy problem with initial time zero for the nonlinear equation (1.19). One then translates the results through the change of variables (1.12) to solve the Cauchy problem with initial time zero for the equation (1.11) or equivalently with infinite initial time for the equation (1.1). The final result can be stated as the following proposition, which is adapted to the equation (1.1) in a neighborhood of infinity in time. We need the notation

$$FH^\rho = \{u \in \mathcal{S}' : F^{-1}u \in H^\rho\} .$$

Proposition 1.1. *Let $1/3 < \gamma < 1/2$ and $2 - 5\gamma/2 < \rho < n/2$. Let $u_0 \in FH^\rho$. Let φ be defined by (1.28) with $\varphi(1) = 0$, with s_0 defined by (1.32) and v_a defined by (1.29) with $v_a(0) = v_0$ (see Lemma 2.5, part (2)). Then there exists $T_\infty > 0$ and there exists a unique solution u of the equation (1.1) such that v_c defined by (1.9) (1.12) or equivalently by (1.13) satisfies $\tilde{v}_c \in \mathcal{C}([T_\infty, \infty), FH^\rho)$ and such that*

$$\tilde{v}_c(t) \rightarrow u_0 \text{ in } FH^\rho \text{ when } t \rightarrow \infty . \quad (1.36)$$

Furthermore $\tilde{u} \in \mathcal{C}([T_\infty, \infty), FH^\rho)$ and \tilde{u} satisfies the estimate

$$\| \tilde{u}(t); FH^\rho \| \leq C a_0 \left(1 + a_0^2(1 + a_0^2)t^{1-\gamma} \right)^{1+[\rho]} \quad (1.37)$$

for all $t \geq T_\infty$, where $[\rho]$ is the integral part of ρ and

$$a_0 = \| u_0; FH^\rho \| .$$

Proposition 1.1 follows from Propositions 4.1 and 4.2 through the change of variables (1.9) or (1.10), which implies in particular that

$$\| \tilde{w}(t); FH^\rho \| = \| \tilde{w}_c(1/t); H^\rho \| = \| w_c(1/t); H^\rho \| . \quad (1.38)$$

As previously mentioned, the condition $\rho > 2 - 5\gamma/2 = 1 - \gamma/2 + 1 - 2\gamma$ is stronger than the subcriticality condition $\rho > 1 - \gamma/2$ for $\gamma < 1/2$. The technical origin of that condition is explained in Remark 3.3 below.

This paper is organized as follows. In Section 2, we introduce some notation and we collect a number of estimates which are used throughout this paper. In Section 3, we study the Cauchy problem for the linearized equation (1.35) with initial time $t_0 \geq 0$. In Section 4, we solve the Cauchy problem with initial time zero for the nonlinear equation (1.19) and we translate the result into the corresponding one for the equation (1.11).

This paper follows I closely and uses the same methods. In order to make it reasonably self contained while avoiding excessive repetition, we have given full statements of the intermediate results, but we have shortened or even omitted some of the proofs when they are identical with those of I.

2 Notation and preliminary estimates

In this section we introduce some notation and we collect a number of estimates which will be used throughout this paper. We denote by $\|\cdot\|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$. For any interval I and any Banach space X we denote by $\mathcal{C}(I, X)$ (resp. $\mathcal{C}_w(I, X)$) the space of strongly (resp. weakly) continuous functions from I to X and by $L^\infty(I, X)$ the space of measurable essentially bounded functions from I to X . For real numbers a and b we use the notation $a \vee b = \text{Max}(a, b)$ and $a \wedge b = \text{Min}(a, b)$. We define $(a)_+ = a \vee 0$ and

$$\begin{aligned} [a]_+ &= (a)_+ && \text{for } a \neq 0 \\ &= \varepsilon \text{ for some } \varepsilon > 0 && \text{for } a = 0. \end{aligned}$$

We shall use the Sobolev spaces \dot{H}_r^σ and H_r^σ defined for $-\infty < \sigma < +\infty$, $1 \leq r < \infty$ by

$$\dot{H}_r^\sigma = \left\{ u : \|u; \dot{H}_r^\sigma\| \equiv \|\omega^\sigma u\|_r < \infty \right\}$$

and

$$H_r^\sigma = \left\{ u : \|u; H_r^\sigma\| \equiv \|<\omega>^\sigma u\|_r < \infty \right\}$$

where $\omega = (-\Delta)^{1/2}$ and $<\cdot> = (1 + |\cdot|^2)^{1/2}$. The subscript r will be omitted both

in H^σ and in the L^r norm if $r = 2$ and we shall use the notation

$$\| \omega^{\sigma \pm 0} u \| = \left(\| \omega^{\sigma + \varepsilon} u \| \| \omega^{\sigma - \varepsilon} u \| \right)^{1/2} \quad \text{for some } \varepsilon > 0 .$$

Note also that for $0 < \gamma < n$ [8]

$$g(u) = \kappa |x|^{-\gamma} \star |u|^2 = \kappa C_{\gamma, n} \omega^{\gamma - n} |u|^2 .$$

We shall use extensively the following Sobolev inequalities.

Lemma 2.1. *Let $1 < q, r < \infty$, $1 < p \leq \infty$ and $0 \leq \sigma < \rho$. If $p = \infty$, assume that $\rho - \sigma > n/r$. Let θ satisfy $\sigma/\rho \leq \theta \leq 1$ and*

$$n/p - \sigma = (1 - \theta)n/q + \theta(n/r - \rho) .$$

Then the following inequality holds

$$\| \omega^\sigma u \|_p \leq C \| u \|_q^{1-\theta} \| \omega^\rho u \|_r^\theta . \quad (2.1)$$

We shall also use extensively the following Leibnitz estimates.

Lemma 2.2. *Let $1 < r, r_1, r_3 < \infty$ and*

$$1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4 .$$

Then the following estimates hold for $\sigma \geq 0$:

$$\| \omega^\sigma(uv) \|_r \leq C (\| \omega^\sigma u \|_{r_1} \| v \|_{r_2} + \| \omega^\sigma v \|_{r_3} \| u \|_{r_4}) . \quad (2.2)$$

An easy consequence of Lemmas 2.1 and 2.2 is the inequality

$$\begin{aligned} \| \omega^\sigma f u \| &\leq C \left(\| f \|_\infty + \| \omega^{n/2} f \| \right) \| \omega^\sigma u \| . \\ &\leq C \| \omega^{n/2 \pm 0} f \| \| \omega^\sigma u \| \end{aligned} \quad (2.3)$$

which holds for $|\sigma| < n/2$.

Another consequence is the following lemma.

Lemma 2.3. *Let $0 < \sigma = \sigma_1 + \sigma_2$ and $\sigma_1 \vee \sigma_2 < n/2$. Then*

$$\| \omega^{\sigma - n/2}(uv) \| \leq C \| \omega^{\sigma_1} u \| \| \omega^{\sigma_2} v \| . \quad (2.4)$$

We shall also need some commutator estimates, which are most conveniently stated in terms of homogeneous Besov spaces $\dot{B}_{r,q}^\sigma$ [1]. In the applications, we shall use only the fact that $\dot{B}_{2,2}^\sigma = \dot{H}^\sigma$.

Lemma 2.4. *Let P_i , $i = 1, 2$ be homogeneous derivative polynomials of degree α_i or ω^{α_i} for $\alpha_i \geq 0$. Let $\lambda > 0$. Then for any (sufficiently regular) functions m , u and v the following estimates hold.*

$$\begin{aligned} | \langle P_1 u, [\omega^\lambda, m] P_2 v \rangle | &\leq C \| m; \dot{B}_{r_0,2}^{\sigma_0} \cap \nabla^{-1} \omega^{1-\nu} L^{q_0} \| \| u; \dot{B}_{r_1,2}^{\sigma_1} \cap L^{q_1} \| \\ &\| v; \dot{B}_{r_2,2}^{\sigma_2} \cap L^{q_2} \| \end{aligned} \quad (2.5)$$

with $0 \leq \nu \leq 1$, $1 \leq r_i, q_i \leq \infty$, $0 \leq i \leq 2$,

$$\delta(q_0) = \sigma_0 + \delta(r_0) - \nu \quad , \quad \delta(q_i) = \sigma_i + \delta(r_i) \quad , \quad i = 1, 2 \quad . \quad (2.6)$$

$$\sum_{0 \leq i \leq 2} \sigma_i + \delta(r_i) = \lambda + \alpha_1 + \alpha_2 + n/2 \quad (2.7)$$

$$\begin{cases} \sigma_0 + (\sigma_1 \wedge \sigma_2) \geq \lambda + \alpha_1 + \alpha_2 \\ \sigma_1 + \sigma_2 \geq \lambda + \alpha_1 + \alpha_2 - \nu \end{cases} \quad (2.8)$$

where $\delta(r) \equiv n/2 - n/r$ and $\nabla^{-1} \omega^{1-\nu} L^q$ is the space of tempered distributions m such that $\omega^{\nu-1} \nabla m \in L^q$.

The proof is given in Appendix A1 of I.

We shall need some estimates of s_0 , v_a and s defined by (1.29) and (1.31)-(1.34). For $0 < \gamma < 1$, $0 < \rho < n/2$ and $\alpha \in \mathbb{R}$, we define

$$\lambda_\alpha = \gamma - (1/2)[\alpha + 1 + \gamma - 2\rho]_+ \quad (2.9)$$

so that λ_α is a decreasing function of α and $\lambda_\alpha \leq \gamma$ in all cases. The subcriticality condition $\rho > 1 - \gamma/2$ is equivalent to $\lambda_1 > 0$. Furthermore, under that condition, $\lambda_0 = \gamma$ for $\gamma < 1/2$ since then $(1 + \gamma)/2 < 1 - \gamma/2$. For clarity, we shall nevertheless keep λ_0 explicitly in that case in some of the estimates.

Lemma 2.5. *Let $0 < \gamma < 1$ and $0 < \rho < n/2$. Let $v_0 \in H^\rho$.*

(1) *Let s_0 be defined by (1.32). Then*

$$\| \omega^{\alpha+n/2 \pm 0} s_0 \| \leq C a_0^2 t^{\lambda_\alpha - 1} \quad (2.10)$$

for $\alpha + 1 + \gamma > 0$, where $a_0 = \|v_0; H^\rho\|$.

Let in addition $\rho > 1 - \gamma/2$. Let $0 < T \leq 1$, $I = (0, T]$ and $\bar{I} = [0, T]$. Then

(2) The equation (1.29) for v_a with initial condition $v_a(0) = v_0$ has a unique solution $v_a \in \mathcal{C}(\bar{I}, H^\rho)$ and that solution satisfies the estimate

$$\|v_a; L^\infty(I, H^\rho)\| \equiv a_a \leq a_0 \exp\left(C a_0^2 T^{\lambda_1}\right). \quad (2.11)$$

Let in addition $\gamma < 1/2$. Then

(3) The following estimates hold for s_b and s_c defined by (1.33) (1.34)

$$\|\omega^{\alpha+n/2\pm 0} s_b\| \leq C a_0^4 (1 - 2\gamma)^{-1} t^{\lambda_0 + \lambda_{\alpha+1} - 1} \quad (2.12)$$

$$\|\omega^{\alpha+n/2\pm 0} s_c\| \leq C a_0^2 a_a^2 \gamma^{-1} (1 - 2\gamma)^{-1} t^{\lambda_0 + \lambda_{\alpha+1} - 1} \quad (2.13)$$

for $\alpha + 2 + \gamma > 0$ and for all $t \in I$.

Sketch of proof.

Part (1) follows from the fact that

$$\begin{aligned} \|\omega^{\alpha+n/2\pm 0} s_0\| &\leq C \int_t^1 dt' t'^{\gamma-2} \|\omega^{\alpha+1+\gamma-n/2\pm 0} \chi_L(t') |v_0|^2\| \\ &\leq C a_0^2 \int_t^1 dt' t'^{\lambda_\alpha-2} \end{aligned} \quad (2.14)$$

by (2.4).

Part (2). The existence of a unique solution v_a of (1.29) as stated can be proved easily, for instance by first solving the Cauchy problem with initial condition $v_a(t_0) = v_0$ for some $t_0 > 0$ by a parabolic regularisation, a fixed point argument and a limiting procedure, and then taking the limit $t_0 \rightarrow 0$ of the solution thereby obtained. The key technical fact consists of preliminary versions of the a priori estimate (2.11), which we now derive. From (1.29), we obtain

$$\begin{aligned} \partial_t \|\omega^\sigma v_a\|^2 &= 2\operatorname{Re} \langle \omega^\sigma v_a, \omega^\sigma (s_0 \cdot \nabla v_a + (1/2)(\nabla \cdot s_0) v_a) \rangle \\ &\leq C \|\omega^{1+n/2\pm 0} s_0(t)\| \|\omega^\sigma v_a\|^2 \\ &\leq C a_0^2 t^{\lambda_1-1} \|\omega^\sigma v_a\|^2 \end{aligned} \quad (2.15)$$

for $0 < \sigma \leq \rho$, by Lemma 2.4 and (2.10), from which (2.11) follows by integration over time.

Part (3). We first estimate s_b . From (1.33) we obtain

$$\begin{aligned} \|\omega^{\alpha+n/2\pm 0} s_b\| &\leq \int_t^1 dt' \|\omega^{\alpha+1+n/2\pm 0} |s_0(t')|^2\| \\ &\leq \int_t^1 dt' \|\omega^{\alpha+1+n/2\pm 0} s_0\| \|\omega^{n/2\pm 0} s_0\| (t') \end{aligned} \quad (2.16)$$

by (2.3), for $\alpha + 1 + n > 0$,

$$\leq C a_0^4 \int_t^1 dt' t'^{\lambda_0+\lambda_{\alpha+1}-2}$$

by (2.10), for $\alpha + 2 + \gamma > 0$,

$$\leq C a_0^4 (1 - \lambda_0 - \lambda_{\alpha+1})^{-1} t^{\lambda_0+\lambda_{\alpha+1}-1} \quad (2.17)$$

from which (2.12) follows since $\lambda_0, \lambda_{\alpha+1} \leq \gamma$. The condition $\alpha + 2 + \gamma > 0$ implies $\alpha + 1 + n > 0$ since $\gamma \leq 1$.

We next estimate s_c . From (1.34) and from the conservation law

$$\partial_t |v_a|^2 = \nabla \cdot s_0 |v_a|^2 \quad (2.18)$$

with $v_a(0) = v_0$, we obtain

$$s_c = - \int_t^1 dt' t'^{\gamma-2} \int_0^{t'} dt'' \nabla \omega^{\gamma-n} \chi_L(t') \nabla \cdot s_0 |v_a|^2(t'') \quad (2.19)$$

so that

$$\begin{aligned} \|\omega^{\alpha+n/2\pm 0} s_c\| &\leq \int_t^1 dt' t'^{\gamma-2} \int_0^{t'} dt'' \|\omega^{\alpha+2+\gamma-n/2\pm 0} \chi_L(t') s_0 |v_a|^2(t'')\| \\ &\leq C \int_t^1 dt' t'^{\lambda_{\alpha+1}-2} \int_0^{t'} dt'' \|\omega^{n/2\pm 0} s_0(t'')\| \|v_a(t''); H^\rho\|^2 \end{aligned} \quad (2.20)$$

by (2.3) (2.4)

$$\begin{aligned} &\leq C a_0^2 a_a^2 \int_t^1 dt' t'^{\lambda_{\alpha+1}-2} \int_0^{t'} dt'' t''^{\lambda_0-1} \\ &\leq C a_0^2 a_a^2 \gamma^{-1} (1 - 2\gamma)^{-1} t^{\lambda_0+\lambda_{\alpha+1}-1} \end{aligned} \quad (2.21)$$

for $\alpha + 2 + \gamma > 0$.

□

In the applications, we shall use (2.12) (2.13) in the form

$$\|\omega^{\alpha+n/2\pm 0} (s - s_0)\| \leq C a_0^2 a_a^2 t^{\lambda_0+\lambda_{\alpha+1}-1} \quad (2.22)$$

where we drop the dependence of the constant on γ .

In order to estimate the term $g_L(v) - g_L(v_a)$ in $L(v)$ (see (1.30)) we shall need estimates of $|v|^2 - |v_a|^2$. For that purpose, we shall use the fact that if v satisfies the equation (1.22) for some real V and if v_a satisfies the equation (1.29), then the following formal conservation law holds

$$\partial_t (|v|^2 - |v_a|^2) = - \operatorname{Im} \bar{v} \Delta v + \nabla \cdot (s|v|^2 - s_0|v_a|^2) \quad (2.23)$$

(compare with (1.25) where v_a satisfied (1.24) instead of (1.29)). We first give sufficient conditions for (2.23) to make sense and preliminary estimates which follow from it. The following lemma is a minor extension of Lemmas 2.6 and 2.7 of I.

Lemma 2.6. *Let $0 < \gamma < 1/2$ and $1 - \gamma/2 < \rho < n/2$. Let $0 < T \leq 1$, $I = (0, T]$ and $\bar{I} = [0, T]$. Let $v_0 \in H^\rho$. Let $v_a \in \mathcal{C}(\bar{I}, H^\rho)$ be the solution of (1.29) with $v_a(0) = v_0$. Let $v \in L^\infty(I, H^\rho) \cap \mathcal{C}(\bar{I}, L^2)$ satisfy the equation (1.22) in I for some real $V \in L^\infty_{loc}(I, L^\infty)$, with $v(0) = v_0$. Then*

$$|v(t)|^2 - |v_a(t)|^2 = V_1(t) + V_2(t) \quad (2.24)$$

where

$$V_1(t) = - \int_0^t dt' \operatorname{Im} \bar{v} \Delta v(t') \quad (2.25)$$

$$V_2(t) = \nabla \cdot \int_0^t dt' (s|v|^2 - s_0|v_a|^2)(t') \quad (2.26)$$

and for all $t \in I$, V_1, V_2 satisfy the estimates

$$\| \omega^{2\sigma-2-n/2} V_1(t) \| \leq C a^2 t \quad (2.27)$$

for $1/2 < \sigma \leq \rho \wedge (1 + n/4)$,

$$\| \omega^{2\sigma-1-n/2} V_2(t) \| \leq C a^2 a_1^2 t^{\lambda_0} (1 + a^2 t^{\lambda_1}) \quad (2.28)$$

for $0 < \sigma \leq \rho$, with

$$a = \| v; L^\infty(I, H^\rho) \|, \quad a_1 = a \exp(C a^2 T^{\lambda_1}).$$

Indication of proof.

The proof is essentially the same as that of Lemma 2.7 of I. In particular the estimate (2.27) is identical with (2.22) of I. Here we give only the proof of the

estimate (2.28), which is new. Note that V_2 here is more complicated than the corresponding V_2 of I. We estimate

$$\begin{aligned} \|\omega^{2\sigma-1-n/2} V_2(t)\| &\leq \int_0^t dt' \|\omega^{2\sigma-n/2} (s|v|^2 - s_0|v_a|^2)\| (t') \\ &\leq C \int_0^t dt' \left(\|\omega^{n/2\pm 0} s\| \|\omega^\sigma v\|^2 + \|\omega^{n/2\pm 0} s_0\| \|\omega^\sigma v_a\|^2 \right) (t') \end{aligned} \quad (2.29)$$

by (2.3) (2.4), from which (2.28) follows by the use of (2.10) (2.22) and from the fact that $a_0 \leq a$ so that $a_a \leq a_1$.

□

The estimate (2.28) of V_2 is too rough for the subsequent applications. In particular it fails to exploit the expected cancellation between $s|v|^2$ and $s_0|v_a|^2$. In order to take the advantage of the latter, we rewrite

$$\begin{aligned} V_2 &= \nabla \cdot \int_0^t dt' ((s - s_0)|v|^2) (t') + \nabla \cdot \int_0^t dt' (s_0 (|v|^2 - |v_a|^2)) (t') \\ &= V_3 + V_3' \end{aligned} \quad (2.30)$$

and we substitute again (2.24) in V_3' so that $V_3' = V_4 + V_5$ with

$$\begin{aligned} V_4 &= \nabla \cdot \int_0^t dt' s_0(t') V_1(t') \\ &= -\nabla \cdot \int_0^t dt' \int_0^{t'} dt'' s_0(t') \operatorname{Im} \bar{v} \Delta v(t'') , \end{aligned} \quad (2.31)$$

$$\begin{aligned} V_5 &= \nabla \cdot \int_0^t dt' s_0(t') V_2(t') \\ &= \nabla \cdot \int_0^t dt' \int_0^{t'} dt'' s_0(t') \nabla \cdot (s|v|^2 - s_0|v_a|^2) (t'') . \end{aligned} \quad (2.32)$$

We then estimate V_3 , V_4 and V_5 in the following lemma.

Lemma 2.7. *Let the assumptions of Lemma 2.6 be satisfied. Then the following estimates hold for all $t \in I$:*

$$\|\omega^{2\sigma-1-n/2} V_3(t)\| \leq C a^4 a_1^2 t^{\lambda_0+\lambda_1} \quad (2.33)$$

for $0 < \sigma \leq \rho$,

$$\begin{aligned} |\langle \psi, V_4(t) \rangle| &\leq C a^4 \left\{ t^{\lambda_0+1} \|\omega^{3-2\sigma+n/2} \psi\| \right. \\ &\quad \left. + \chi(\sigma \leq 1) t^{\lambda_*+1} \|\omega^{1+n/2\pm 0} \psi\| \right\} \end{aligned} \quad (2.34)$$

for $1/2 < \sigma \leq \rho \wedge (1 + n/4)$ and for all ψ such that the last two norms are finite whenever they occur, with

$$\lambda_\star = \gamma - (1/2)(3 + \gamma - 2\sigma - 2\rho)_+ \quad (2.35)$$

and $\chi(\sigma \leq 1) = 1$ (resp. 0) if $\sigma \leq 1$ (resp. $\sigma > 1$).

$$\| \omega^{2\sigma-2-n/2} V_5(t) \| \leq C a^4 a_1^2 t^{2\lambda_0} (1 + a^2 t^{\lambda_1}) \quad (2.36)$$

for $1/2 < \sigma \leq \rho$.

Proof. We first estimate V_3 . We obtain

$$\begin{aligned} \| \omega^{2\sigma-1-n/2} V_3(t) \| &\leq \int_0^t dt' \| \omega^{2\sigma-n/2}(s-s_0)|v|^2 \| (t') \\ &\leq C \int_0^t dt' \| \omega^{n/2\pm 0}(s-s_0) \| \| \omega^\sigma v \|^2 (t') \end{aligned} \quad (2.37)$$

by (2.3) (2.4), from which (2.33) follows by the use of (2.22) and integration over time.

We next estimate V_4 . Let $w = \text{Im } \bar{v}\Delta v$. We rewrite

$$\langle \psi, V_4 \rangle = \int_0^t dt' \int_0^{t'} dt'' \langle s_0(t') \cdot \nabla \psi, w(t'') \rangle \quad (2.38)$$

and we know (see the proof of (2.22) in Lemma 2.7 of I) that

$$\| \omega^{2\sigma-2-n/2} w \| \leq C \| \omega^\sigma v \|^2 \quad (2.39)$$

for $1/2 < \sigma \leq \rho \wedge (1 + n/4)$, so that

$$|\langle \psi, V_4 \rangle| \leq C \int_0^t dt' \int_0^{t'} dt'' \| \omega^{n/2+2-2\sigma} s_0(t') \nabla \psi \| \| \omega^\sigma v(t'') \|^2. \quad (2.40)$$

We next estimate

$$\begin{aligned} \| \omega^{n/2+2-2\sigma} s_0(t') \nabla \psi \| &\leq C \left\{ \| \omega^{n/2\pm 0} s_0(t') \| \| \omega^{n/2+3-2\sigma} \psi \| \right. \\ &\quad \left. + \chi(\sigma \leq 1) \| \omega^{n/2+2-2\sigma} s_0(t') \| \| \omega^{1+n/2\pm 0} \psi \| \right\} \end{aligned} \quad (2.41)$$

by (2.3) (2.4). The estimate (2.34) then follows from (2.40) (2.41) by the use of (2.10) and integration over time. The condition $3 - 2\sigma + \gamma > 0$ needed to apply (2.10) in the second term of (2.41) is always fulfilled for $\sigma \leq 1$.

We finally estimate V_5 . From (2.32) we obtain

$$\begin{aligned} \|\omega^{2\sigma-2-n/2} V_5\| &\leq \int_0^t dt' \|\omega^{2\sigma-1-n/2} s_0(t') V_2(t')\| \\ &\leq C \int_0^t dt' \|\omega^{n/2\pm 0} s_0(t')\| \|\omega^{2\sigma-1-n/2} V_2(t')\| \end{aligned} \quad (2.42)$$

for $1/2 < \sigma \leq \rho$ by (2.3), and (2.36) follows from (2.42) by the use of (2.10) (2.28) and integration over time.

□

For $\sigma > 1$, the estimate (2.34) of V_4 reduces to

$$\|\omega^{2\sigma-3-n/2} V_4\| \leq C a^4 t^{\lambda_0+1}. \quad (2.43)$$

In the more interesting case $\sigma \leq 1$, it yields an estimate of V_4 in the space $\dot{H}^{2\sigma-3-n/2} + \dot{H}^{-1-n/2\pm 0}$, but that space is not space dilation homogeneous. In the applications, we shall use (2.34) with a time dependent ψ , which will restore the space time dilation homogeneity of the estimate. More generally, we shall use the following lemma.

Lemma 2.8. *Let the assumptions of Lemma 2.6 be satisfied.*

(1) *Let $0 < \sigma' < 1 + \gamma$ and let*

$$\mu_j = \gamma - (1/2)(j + 1 + \gamma - \sigma' - 2\rho)_+ . \quad (2.44)$$

Then the following estimates hold :

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_1(t)\| \leq C a^2 t^{\mu_1-1}, \quad (2.45)$$

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_3(t)\| \leq C a^4 a_1^2 t^{2\lambda_0+\mu_1-2}, \quad (2.46)$$

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_4(t)\| \leq C a^4 t^{\lambda_0+\mu_2-1}, \quad (2.47)$$

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_5(t)\| \leq C a^4 a_1^2 t^{2\lambda_0+\mu_1-2} (1 + a^2 t^{\lambda_1}). \quad (2.48)$$

(2) *Let $\sigma' = 0$. Then the estimates (2.45)-(2.48) hold with μ_j replaced by λ_j in the right hand sides.*

Proof Part (1). We first estimate V_1 . From (2.27) we obtain

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_1\| \leq C a^2 t^{\gamma-1-(1/2)(2+\gamma-\sigma'-2\sigma)} \quad (2.49)$$

under the conditions

$$\begin{cases} 1 < 2\sigma \leq 2\rho \wedge (2 + n/2) \\ \sigma' + 2\sigma \leq 2 + \gamma \end{cases} \quad (2.50)$$

(which make the condition $\sigma' < 1 + \gamma$ unavoidable).

For $\sigma' + 2\rho \geq 2 + \gamma$, we choose σ so that $\sigma' + 2\sigma = 2 + \gamma$.

For $\sigma' + 2\rho \leq 2 + \gamma$, we choose $\sigma = \rho$. That choice satisfies (2.50). In particular, the condition $2\sigma \leq 2 + n/2$ follows from the condition $2\sigma \leq 2 + \gamma$. This proves (2.45).

We next estimate V_3 . The estimate (2.33) is not sufficient for that purpose. We estimate instead for $\sigma' < 2\sigma$

$$\begin{aligned} \|\omega^{2\sigma-\sigma'-1-n/2} V_3(t)\| &\leq \int_0^t dt' \|\omega^{2\sigma-\sigma'-n/2} (s-s_0)|v|^2\| (t') \\ &\leq C \int_0^t dt' \|\omega^{n/2-\sigma'}(s-s_0)\| \|\omega^\sigma v\|^2 (t') \end{aligned}$$

by (2.4),

$$\leq C a^4 a_1^2 t^{\lambda_0+\mu_1} \quad (2.51)$$

by (2.22) (we do not need the L^∞ norm of $s-s_0$, which allows us to use μ_j defined by (2.44) with $(\)_+$ instead of $[\]_+$). From (2.51) we obtain

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_3\| \leq C a^4 a_1^2 t^{\lambda_0+\mu_1+\gamma-(1/2)(1+\gamma-2\sigma)-2}$$

for $0 < \sigma \leq \rho \wedge (1 + \gamma)/2$. Choosing $\sigma = \rho \wedge (1 + \gamma)/2$ yields (2.46).

We next estimate V_4 . From (2.34) we obtain

$$\begin{aligned} t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_4\| &\leq C a^4 \sup_{\|\psi\|=1} t^{\gamma-1} \\ &\quad \left\{ t^{\lambda_0} \|\omega^{3+\gamma-\sigma'-2\sigma} \chi_L \psi\| + \chi(\sigma \leq 1) t^{\lambda_*} \|\omega^{1+\gamma-\sigma'\pm 0} \chi_L \psi\| \right\} \\ &\leq C a^4 \left\{ t^{\lambda_0-1+\gamma-(1/2)(3+\gamma-\sigma'-2\sigma)} \right. \\ &\quad \left. + \chi(\sigma \leq 1) t^{\lambda_*-1+\gamma-(1/2)(1+\gamma-\sigma')} \right\} \end{aligned} \quad (2.52)$$

under the conditions

$$\begin{cases} 1 < 2\sigma \leq 2\rho \wedge (2 + n/2) \\ \sigma' + 2\sigma \leq 3 + \gamma \end{cases} \quad (2.53)$$

(together with the condition $\sigma' < 1 + \gamma$). We next show that the second term in the last bracket in (2.52) is better behaved, namely has a larger time exponent than the first one. In fact

$$\begin{aligned} &2(\lambda_* + \gamma - (1/2)(1 + \gamma - \sigma') - \lambda_0) - 2\gamma + (3 + \gamma - \sigma' - 2\sigma) \\ &= -(3 + \gamma - 2\sigma - 2\rho)_+ + 2 - 2\sigma + (1 + \gamma - 2\rho)_+ \geq 0 \end{aligned}$$

for $\sigma \leq 1$, since $(a+b)_+ \leq a_+ + b$ for $b \geq 0$. We can therefore omit that second term. We next choose σ .

For $\sigma' + 2\rho \geq 3 + \gamma$, we choose σ so that $\sigma' + 2\sigma = 3 + \gamma$.

For $\sigma' + 2\rho \leq 3 + \gamma$, we choose $\sigma = \rho$. That choice satisfies (2.53). In particular the condition $2\sigma > 1$ follows from $\sigma' < 1 + \gamma$ in the first case and from $2\rho > 1$ in the second one. The condition $2\sigma \leq 2 + n/2$ follows from $2\sigma \leq 3 + \gamma$ for $n \geq 3$ and from $2\sigma \leq 2\rho < n$ for $n \leq 4$. This proves (2.47).

We finally estimate V_5 . From (2.36) we obtain

$$t^{\gamma-2} \|\omega^{\gamma-\sigma'-n/2} \chi_L V_5\| \leq C a^4 a_1^2 (1 + a^2 t^{\lambda_1}) t^{2\lambda_0+\gamma-2-1/2(2+\gamma-\sigma'-2\sigma)} \quad (2.54)$$

under the conditions

$$\begin{cases} 1 < 2\sigma \leq 2\rho \\ \sigma' + 2\sigma \leq 2 + \gamma. \end{cases} \quad (2.55)$$

For $\sigma' + 2\rho \geq 2 + \gamma$, we choose σ so that $\sigma' + 2\sigma = 2 + \gamma$.

For $\sigma' + 2\rho \leq 2 + \gamma$, we choose $\sigma = \rho$. That choice satisfies (2.55). In particular the condition $2\sigma > 1$ follows from $\sigma' < 1 + \gamma$ in the first case and from $2\rho > 1$ in the second one. This proves (2.48).

Part (2). The proof is similar but simpler.

□

Remark 2.1. The fact that $\sigma' = 0$ in the proof of Part (2) and more generally the need of L^∞ norms requires the use of $[\]_+$ in λ_j , whereas for $\sigma' > 0$ one can use $(\)_+$ in the definition of μ_j .

3 The linearized Cauchy problem for \mathbf{v}

In this section we study the Cauchy problem for the linearized equation (1.35) with $L(v)$ defined by (1.30) for a given v , with initial time $t_0 \geq 0$. We first give a preliminary result with $t_0 > 0$, where we do not study the behaviour of the solution as t tends to zero.

Proposition 3.1. *Let $0 < \gamma < 1$ and $\rho > 1 - \gamma/2$. Let $I = (0, T]$, let $v_0 \in H^\rho$ and let $v \in L_{loc}^\infty(I, H^\rho)$. Let s , s_0 and v_a be defined by (1.31)-(1.34) and (1.29) with $v_a(0) = v_0$. Let $0 \leq \rho' < n/2$, let $0 < t_0 \leq T$ and let $v'_0 \in H^{\rho'}$. Then the equation*

(1.35) has a unique solution $v' \in \mathcal{C}(I, H^{\rho'})$ with $v'(t_0) = v'_0$. The solution satisfies

$$\|v'(t)\| = \|v'_0\|$$

for all $t \in I$ and is unique in $\mathcal{C}(I, L^2)$.

The proof is sketched in Appendix A2 of I.

We next study the boundedness and continuity properties near $t = 0$ of the solutions of (1.35) obtained in Proposition 3.1. Since we shall eventually be interested in taking $\rho' = \rho$, we already impose the condition $\rho < n/2$ in the next proposition (see however Remark 3.2 below).

Proposition 3.2. *Let $1/3 < \gamma < 1/2$ and $2 - 5\gamma/2 < \rho < n/2$. Let $I = (0, T]$ and $\bar{I} = [0, T]$, let $v_0 \in H^\rho$ and let $v \in L^\infty(I, H^\rho) \cap \mathcal{C}(\bar{I}, L^2)$ with $v(0) = v_0$. Let s, s_0 and v_a be defined by (1.31)-(1.34) and (1.29) with $v_a(0) = v_0$. Let v satisfy the equation (1.22) in I for some real $V \in L^\infty_{loc}(I, L^\infty)$. Let $1/2 \leq \rho' < n/2$ and let $v' \in \mathcal{C}(I, H^{\rho'})$ be a solution of the equation (1.35) in I . Then*

- (1) $v' \in (\mathcal{C} \cap L^\infty)(I, H^{\rho'}) \cap \mathcal{C}_w(\bar{I}, H^{\rho'}) \cap \mathcal{C}(\bar{I}, H^\sigma)$ for $0 \leq \sigma < \rho'$.
- (2) For all $t \in \bar{I}$, $t_1 \in I$, the following estimate holds

$$\|\omega^{\rho'} v'(t)\| \leq \|\omega^{\rho'} v'(t_1)\| E(|t - t_1|) \quad (3.1)$$

where

$$E(t) \equiv E(t, a) = \exp \left\{ C a^2 (1 + a^2) (1 + a_1^2)^2 t^{2\gamma + \lambda_1 - 1} \right\}, \quad (3.2)$$

$$a = \|v; L^\infty(I, H^\rho)\|, \quad a_1 = a \exp \left(C a^2 T^{\lambda_1} \right) \quad (3.3)$$

and λ_j is defined by (2.9).

- (3) For all $t, t_1 \in \bar{I}$, the following estimate holds

$$\|v'(t) - v'(t_1)\| \leq C |t - t_1|^{\rho' \gamma \wedge (3\gamma - 1)} (1 + a^2)^2 (1 + a_1^2)^2 \|v'(t_1); H^{\rho'}\|. \quad (3.4)$$

Remark 3.1. The estimate (3.1) for $t, t_1 \in I$ holds for $0 \leq \rho' < n/2$, as will be clear from the proof. The condition $\rho' \geq 1/2$ is needed to derive (3.4) which is used in turn to extend (3.1) to $t = 0$.

Remark 3.2. The assumption $\rho < n/2$ in Proposition 3.2 can be dispensed with at the expense of using slightly different estimates, which yield different powers of t

in (3.2) and (3.4).

Proof. We know already that the L^2 - norm of v' is conserved. The bulk of the proof consists in deriving the estimates (3.1) and (3.4) for $t, t_1 \in I$. We begin with (3.1). From (1.30) (1.35) we obtain

$$\begin{aligned} \partial_t \|\omega^{\rho'} v'\|^2 &= \operatorname{Im} \langle v', [\omega^{2\rho'}, L(v)]v' \rangle \\ &= \operatorname{Re} \langle v', [\omega^{2\rho'}, s] \cdot \nabla v' \rangle + \operatorname{Im} \langle v', [\omega^{2\rho'}, f]v' \rangle \end{aligned} \quad (3.5)$$

where

$$f = (1/2) (|s|^2 - |s_0|^2) + t^{\gamma-2} g_S(v) + t^{\gamma-2} (g_L(v) - g_L(v_a)) . \quad (3.6)$$

We estimate the first term in the RHS of (3.5) by Lemma 2.4 with $\lambda = 2\rho'$, $\alpha_1 = 0$, $\alpha_2 = 1$, $r_i = 2$, $\sigma_1 = \sigma_2 = \rho'$ and $\nu = 1$, so that $\sigma_0 = 1 + n/2$ and $q_0 = \infty$.

We estimate similarly the last term by Lemma 2.4 with $\lambda = 2\rho'$, $\alpha_1 = \alpha_2 = 0$, $r_i = 2$, $\sigma_1 = \sigma_2 = \rho'$ and $\nu = 1$, so that $\sigma_0 = n/2$ and $\delta(q_0) = n/2 - 1$. We obtain

$$\left| \partial_t \|\omega^{\rho'} v'\|^2 \right| \leq C \left(\|\omega^{n/2} \nabla s\| + \|\nabla s\|_\infty + \|\omega^{n/2} f\| \right) \|\omega^{\rho'} v'\|^2 . \quad (3.7)$$

We estimate the various norms successively. We first estimate ∇s by (2.10) (2.22) with $\alpha = 1$ so that

$$\|\omega^{n/2} \nabla s\| + \|\nabla s\|_\infty \leq C \|\omega^{n/2+1\pm 0} s\| \leq C a^2 \{t^{\lambda_1-1} + a_1^2 t^{\lambda_0+\lambda_2-1}\} . \quad (3.8)$$

We next estimate the contribution of f . From (2.10) (2.22) with $\alpha = 0$, we obtain

$$\begin{aligned} \|\omega^{n/2} (|s|^2 - |s_0|^2)\| &\leq C \|\omega^{n/2\pm 0} (s + s_0)\| \|\omega^{n/2\pm 0} (s - s_0)\| \\ &\leq C a^4 a_1^2 t^{2\lambda_0+\lambda_1-2} (1 + a_1^2 t^{\lambda_1}) . \end{aligned} \quad (3.9)$$

We next estimate

$$\begin{aligned} t^{\gamma-2} \|\omega^{n/2} g_S(v)\| &\leq C t^{\gamma-2+\rho-\gamma/2} \|\omega^\rho v\|^2 \\ &\leq C a^2 t^{\lambda_1-1} \end{aligned} \quad (3.10)$$

for $\rho \geq \gamma/2$.

The contribution of the last term in f is estimated by the use of Lemma 2.8, part (2). We obtain

$$\begin{aligned} t^{\gamma-2} \|\omega^{n/2} (g_L(v) - g_L(v_a))\| &= C t^{\gamma-2} \|\omega^{\gamma-n/2} \chi_L (V_1 + V_3 + V_4 + V_5)\| \\ &\leq C a^2 \{t^{\lambda_1-1} + a^2 t^{\lambda_0+\lambda_2-1} + a^2 a_1^2 t^{2\lambda_0+\lambda_1-2} (1 + a^2 t^{\lambda_1})\} . \end{aligned} \quad (3.11)$$

Collecting (3.7)-(3.11), we obtain

$$\left| \partial_t \parallel \omega^{\rho'} v'(t) \parallel^2 \right| \leq N(t) \parallel \omega^{\rho'} v'(t) \parallel^2 \quad (3.12)$$

where

$$N(t) = C a^2 \left\{ t^{\lambda_1-1} + a_1^2 t^{\lambda_0+\lambda_2-1} + a^2 a_1^2 t^{2\lambda_0+\lambda_1-2} (1 + a_1^2 t^{\lambda_1}) \right\} . \quad (3.13)$$

In order to estimate $\parallel \omega^{\rho'} v'(t) \parallel$, we need $N(t)$ to be integrable in time at $t = 0$. We first compare the various time exponents occurring in (3.13), assuming only that $0 < \gamma < 1/2$ and $\rho > 1 - \gamma/2$, which is equivalent to $\lambda_1 > 0$. Clearly

$$2\lambda_0 + \lambda_1 - 1 < \lambda_1 \wedge (2\lambda_0 + 2\lambda_1 - 1) .$$

Moreover, from $[a + b]_+ \leq [a_+]_+ + b$ for $b > 0$, it follows that

$$[3 + \gamma - 2\rho]_+ \leq [2 + \gamma - 2\rho]_+ + 2 - 2\gamma \quad (3.14)$$

and therefore $\lambda_0 + \lambda_2 \geq 2\lambda_0 + \lambda_1 - 1$. Keeping in (3.13) the dominant power of t and using the fact that $\lambda_0 = \gamma$ under the previous assumptions, we obtain

$$N(t) \leq C a^2 (1 + a^2) (1 + a_1^2)^2 t^{2\gamma+\lambda_1-2} . \quad (3.15)$$

The integrability condition of $N(t)$ at $t = 0$ then becomes $2\gamma + \lambda_1 - 1 > 0$ or equivalently $\gamma > 1/3$ and $\rho > 2 - 5\gamma/2$.

The estimate (3.1) (3.2) for $t_1, t \in I$ follows from (3.12) (3.15) by integration.

We next derive the estimate (3.4) for $t, t_1 \in I$. For that purpose we define (see (1.8))

$$\tilde{v}'(t) = U(-t)v'(t) \quad (3.16)$$

$$\tilde{L} = L(v) + (1/2)\Delta = is \cdot \nabla + (i/2)(\nabla \cdot s) + f \quad (3.17)$$

with f given by (3.6). We rewrite (1.35) as

$$i\partial_t \tilde{v}' = U(-t)\tilde{L}U(t)\tilde{v}' \quad (3.18)$$

so that for $t, t_1 \in I$, for fixed t_1 ,

$$\begin{aligned} \partial_t \parallel \tilde{v}'(t) - \tilde{v}'(t_1) \parallel^2 &= 2 \operatorname{Im} \langle \tilde{v}'(t) - \tilde{v}'(t_1), U(-t)\tilde{L}U(t)\tilde{v}'(t_1) \rangle \\ &= 2 \operatorname{Im} \langle w, \tilde{L}v_\star \rangle \end{aligned} \quad (3.19)$$

where

$$\begin{cases} v_*(t) = U(t - t_1)v'(t_1) \\ w(t) = v'(t) - v_*(t) . \end{cases} \quad (3.20)$$

We estimate

$$\begin{aligned} |\partial_t \|w\|^2| &\leq 2 |\operatorname{Re} \langle w, s \cdot \nabla v_* \rangle| \\ &+ C \|w\| \left(\|\omega^{n/2-\sigma'} \nabla \cdot s\| + \|\omega^{n/2-\sigma'} f\| \right) \|\omega^{\sigma'} v'(t_1)\| \end{aligned} \quad (3.21)$$

for some σ' with $0 < \sigma' \leq \rho'$, to be chosen later.

For $0 < \rho' < 1$, we write

$$\langle w, s \cdot \nabla v_* \rangle = - \langle \omega^{-\rho'} \nabla \cdot s w, \omega^{\rho'} v_* \rangle \quad (3.22)$$

and we estimate by Lemma 2.2

$$|\langle w, s \cdot \nabla v_* \rangle| \leq C \|\omega^{1-\rho'} w\| \left(\|s\|_\infty + \|\omega^{n/2} s\| \right) \|\omega^{\rho'} v'(t_1)\|. \quad (3.23)$$

For $\rho' = 1$, we estimate

$$|\langle w, s \cdot \nabla v_* \rangle| \leq \|w\| \|s\|_\infty \|\omega^{\rho'} v'(t_1)\|. \quad (3.24)$$

For $\rho' > 1$, we estimate

$$|\langle w, s \cdot \nabla v_* \rangle| \leq C \|w\| \|\omega^{n/2-\sigma'} \nabla s\| \|\omega^{\sigma'} v'(t_1)\| \quad (3.25)$$

for $1 < \sigma' \leq \rho'$.

Collecting (3.21)-(3.25) yields

$$\begin{aligned} |\partial_t \|w\|^2| &\leq C \left\{ \chi(\rho' \leq 1) \|\omega^{1-\rho'} w\| \left(\|s\|_\infty + \|\omega^{n/2} s\| \right) \|\omega^{\rho'} v'(t_1)\| \right. \\ &\left. + \|w\| \left(\|\omega^{n/2-\sigma'} \nabla s\| + \|\omega^{n/2-\sigma'} f\| \right) \|\omega^{\sigma'} v'(t_1)\| \right\} \end{aligned} \quad (3.26)$$

with $0 < \sigma' \leq \rho'$ and $\sigma' > 1$ in the ∇s term if $\rho' > 1$.

For $1/2 \leq \rho' \leq 1$, we interpolate

$$\|\omega^{1-\rho'} w\| \leq y^\theta \|\omega^{\rho'} w\|^{1/\rho'-1}$$

where

$$y = \|w\|^2, \quad \theta = 1 - 1/(2\rho')$$

so that (3.26) becomes

$$\begin{aligned} |\partial_t y| \leq & C \left\{ \chi(\rho' \leq 1) \left(\|s\|_\infty + \|\omega^{n/2} s\| \right) a_1'^{1/\rho'} y^\theta \right. \\ & \left. + \left(\|\omega^{n/2-\sigma'} \nabla s\| + \|\omega^{n/2-\sigma'} f\| \right) a_1' y^{1/2} \right\} \end{aligned} \quad (3.27)$$

with $a_1' = \|v'(t_1); H^{\rho'}\|$. We estimate the various norms in (3.27) successively. From (1.31) (2.10) (2.22) with $\alpha = 0$ we obtain

$$\|s\|_\infty + \|\omega^{n/2} s\| \leq C a^2 t^{\lambda_0-1} (1 + a_1^2 t^{\lambda_1}) \quad (3.28)$$

and with $\alpha = 1 - \sigma'$, we obtain

$$\|\omega^{n/2-\sigma'} \nabla s\| \leq C a^2 (t^{\mu_1-1} + a_1^2 t^{\lambda_0+\mu_2-1}) \quad (3.29)$$

for $0 < \sigma' < 2 + \gamma$ (see Remark 2.1).

We next estimate the contribution of f . From (2.10) (2.22) we obtain

$$\begin{aligned} \|\omega^{n/2-\sigma'} (|s|^2 - |s_0|^2)\| &\leq C \|\omega^{n/2\pm 0}(s + s_0)\| \|\omega^{n/2-\sigma'}(s - s_0)\| \\ &\leq C a^4 a_1^2 t^{2\lambda_0+\mu_1-2} (1 + a_1^2 t^{\lambda_1}) \end{aligned} \quad (3.30)$$

for $0 < \sigma' < (2 + \gamma) \wedge n$. We then estimate

$$t^{\gamma-2} \|\omega^{n/2-\sigma'} g_S(v)\| \leq C a^2 t^{\mu_1-1}, \quad (3.31)$$

for $0 < \sigma' \leq 2\rho - \gamma$.

The contribution of the last term in f is estimated by the use of Lemma 2.8, part (1) for $0 < \sigma' < 1 + \gamma$ and $\sigma' \leq \rho'$:

$$\begin{aligned} t^{\gamma-2} \|\omega^{n/2-\sigma'} (g_L(v) - g_L(v_a))\| &= C t^{\gamma-2} \|\omega^{\gamma-n/2-\sigma'} \chi_L(V_1 + V_3 + V_4 + V_5)\| \\ &\leq C a^2 \left\{ t^{\mu_1-1} + a^2 t^{\lambda_0+\mu_2-1} + a^2 a_1^2 t^{2\lambda_0+\mu_1-2} (1 + a^2 t^{\lambda_1}) \right\}. \end{aligned} \quad (3.32)$$

Collecting (3.27)-(3.32), we obtain

$$\begin{aligned} |\partial_t y| \leq & C \left\{ \chi(\rho' \leq 1) a^2 t^{\lambda_0-1} (1 + a_1^2 t^{\lambda_1}) a_1'^{1/\rho'} y^\theta \right. \\ & \left. + a^2 (t^{\mu_1-1} + a_1^2 t^{\lambda_0+\mu_2-1} + a^2 a_1^2 t^{2\lambda_0+\mu_1-2} (1 + a_1^2 t^{\lambda_1})) a_1' y^{1/2} \right\}. \end{aligned} \quad (3.33)$$

We next choose σ' as large as possible, namely $\sigma' = \rho' \wedge (1 + \gamma - 0)$ and we compare the various time exponents that occur in the last term of (3.33), assuming only that $0 < \gamma < 1/2$ and $\rho > 1 - \gamma/2$, which is equivalent to $\lambda_1 > 0$. Clearly

$2\lambda_0 + \mu_1 - 1 < \mu_1 \wedge (2\lambda_0 + \lambda_1 + \mu_1 - 1)$. We next show that $\lambda_0 + \mu_2 - 1 \geq 2\lambda_0 + \mu_1 - 2$, or equivalently

$$(3 + \gamma - 2\rho - \rho' \wedge (1 + \gamma - 0))_+ \leq (2 + \gamma - 2\rho - \rho')_+ + 2(1 - \gamma) , \quad (3.34)$$

where we have used the fact that the limitation $\sigma' < 1 + \gamma$ is not seen in μ_1 for $\rho > 1/2$. The inequality (3.34) with ρ' is proved in the same way as (3.14) and the inequality with $1 + \gamma$ is trivial since

$$2 - 2\rho < \gamma \leq 2(1 - \gamma) .$$

The dominant exponent is then

$$\begin{aligned} 2\lambda_0 + \mu_1 - 1 &= 3\gamma - 1 - (1/2)(2 + \gamma - 2\rho - \rho')_+ \\ &= (3\gamma - 1) \wedge (\rho - 2 + 5\gamma/2 + \rho'/2) \\ &\geq \mu \equiv (3\gamma - 1) \wedge \rho'/2 \end{aligned}$$

for $\rho > 2 - 5\gamma/2$.

From (3.33) we then obtain

$$\begin{aligned} |\partial_t y| &\leq C \left\{ \chi(\rho' \leq 1) a^2 (1 + a_1^2) t^{\gamma-1} a_1'^{1/\rho'} y^\theta \right. \\ &\quad \left. + a^2(1 + a^2) (1 + a_1^2)^2 t^{\mu-1} a_1' y^{1/2} \right\} . \end{aligned} \quad (3.35)$$

Using the fact that the differential inequality

$$|\partial_t y| \leq \sum_i b_i t^{\nu_i-1} y^{\theta_i}$$

with $0 \leq \theta_i < 1$ and $\nu_i > 0$ implies

$$y(t) \leq C \sum_i \left(b_i \nu_i^{-1} |t^{\nu_i} - t_1^{\nu_i}| \right)^{1/(1-\theta_i)}$$

for $t, t_1 > 0$ and $y(t_1) = 0$, we obtain

$$y(t) \leq C \left\{ \chi(\rho' \leq 1) \left(a^2(1 + a_1^2) \right)^{2\rho'} |t^\gamma - t_1^\gamma|^{2\rho'} + \left(a^2(1 + a^2)(1 + a_1^2)^2 \right)^2 |t^\mu - t_1^\mu|^2 \right\} a_1'^2 \quad (3.36)$$

so that

$$\| w(t) \| \leq C(1 + a^2)^2(1 + a_1^2)^2 \left\{ \chi(\rho' \leq 1) |t - t_1|^{\gamma\rho'} + |t - t_1|^\mu \right\} a_1' . \quad (3.37)$$

On the other hand

$$\begin{aligned} \|v'(t) - v'(t_1)\| &\leq \|w(t)\| + \|(U(t - t_1) - 1)v'(t_1)\| \\ &\leq \|w(t)\| + |t - t_1|^{(\rho'/2) \wedge 1} \|\omega^{\rho' \wedge 2} v'(t_1)\|. \end{aligned} \quad (3.38)$$

Collecting (3.37) (3.38) yields (3.4) for $t, t_1 \in I$.

We now exploit (3.1) and (3.4) in I to complete the proof of the proposition. From (3.1) it follows that $v' \in L^\infty(I, H^{\rho'})$. From (3.1) and (3.4) it then follows that v' has a limit $v'(0)$ in L^2 and that (3.4) holds for $t, t_1 \in \bar{I}$. It then follows by a standard abstract argument that $v'(0) \in H^{\rho'}$, that $v' \in \mathcal{C}_w(\bar{I}, H^{\rho'}) \cap \mathcal{C}(\bar{I}, H^\sigma)$ for $0 \leq \sigma < \rho'$, and that (3.1) holds for all $t \in \bar{I}, t_1 \in I$.

□

Remark 3.3. The integrability of $N(t)$ at $t = 0$ requires stronger conditions on ρ than the subcriticality condition $\rho > 1 - \gamma/2$, or equivalently $\lambda_1 > 0$. That condition suffices to control the contributions of ∇s_0 , of $g_S(v)$ and of V_1 to (3.7). The terms $\nabla(s - s_0)$ and V_4 require $\lambda_0 + \lambda_2 > 0$, or equivalently $\rho > 3/2 - 3\gamma/2 = 1 - \gamma/2 + (1/2 - \gamma)$. The worst terms are $|s|^2 - |s_0|^2$, V_3 and V_5 which require $2\lambda_0 + \lambda_1 > 1$, or equivalently $\gamma > 1/3$ and $\rho > 2 - 5\gamma/2 = 1 - \gamma/2 + (1 - 2\gamma)$.

We have not proved so far that $v' \in \mathcal{C}(\bar{I}, H^{\rho'})$. This is true but requires a separate argument.

Proposition 3.3. *Under the assumptions of Proposition 3.2, $v' \in \mathcal{C}(\bar{I}, H^{\rho'})$ and (3.1) holds for all $t, t_1 \in \bar{I}$.*

The proof is identical with that of Proposition 3.3 of [4].

We can now state the main result on the Cauchy problem for the linearized equation (1.35).

Proposition 3.4. *Let $1/3 < \gamma < 1/2$ and $2 - 5\gamma/2 < \rho < n/2$. Let $I = (0, T]$ and $\bar{I} = [0, T]$ and let $v \in L^\infty(I, H^\rho) \cap \mathcal{C}(\bar{I}, L^2)$ with $v(0) = v_0$. Let s, s_0 and v_a be defined by (1.31)-(1.34) and (1.29) with $v_a(0) = v_0$. Let v satisfy the equation (1.22) in I for some real $V \in L^\infty_{loc}(I, L^\infty)$. Let $1/2 < \rho' < n/2$ and let $v'_0 \in H^{\rho'}$. Let $t_0 \in \bar{I}$. Then there exists a unique solution $v' \in \mathcal{C}(\bar{I}, H^{\rho'})$ of the equation (1.35) with $v'(0) = v'_0$. Furthermore v' satisfies the estimates (3.1) and (3.4) for all $t, t_1 \in \bar{I}$.*

The solution is actually unique in $\mathcal{C}(\bar{I}, L^2)$.

The proof is identical with that of Proposition 3.4 of I.

4 The nonlinear Cauchy problem at time zero for v and u_c

In this section we prove that the nonlinear equation (1.19) for v , with $L(v)$ defined by (1.30), with initial data at time zero, has a unique solution in a small time interval. We then rewrite that result in terms of u_c , related to v by (1.12), and we give some additional bounds and regularity properties for u_c . In order to solve the equation (1.19) for v , we show that the map $\Gamma : v \rightarrow v'$ defined by Proposition 3.4 with $t_0 = 0$ is a contraction. For that purpose, we need to estimate the difference of two solutions of the linearized equation (1.35). For any pair of functions or operators (f_1, f_2) , we define

$$f_{\pm} = (1/2) (f_2 \pm f_1) .$$

Lemma 4.1. *Let $1/3 < \gamma < 1/2$ and $2 - 5\gamma/2 < \rho < n/2$. Let $I = (0, T]$ and let v_i , $i = 1, 2$ satisfy the assumptions of Proposition 3.4 with $v_i(0) = v_0$. Let $1/2 < \rho' < n/2$ and let v'_i , $i = 1, 2$ be the solutions of the equation (1.35) with $v'_i(0) = v'_0 \in H^{\rho'}$ obtained in Proposition 3.4. Then the following estimate holds for all $t \in I$:*

$$\| v'_-; L^\infty((0, t], H^{\rho'}) \| \leq C E(t, a) a(1+a^2)^2 (1 + a_1^2) a' t^{2\gamma+\lambda_1-1} \| v_-; L^\infty((0, t], H^\rho) \| \quad (4.1)$$

where $E(t, a)$ is defined by (3.2) and

$$a = \text{Max} \| v_i; L^\infty(I, H^\rho) \| , \quad a' = \text{Max} \| v'_i; L^\infty(I, H^{\rho'}) \| . \quad (4.2)$$

Proof. From (1.35) we obtain

$$i\partial_t v'_- = L_2 v'_- + L_- v'_1$$

where $L_i = L(v_i)$, $g_i = g(v_i)$, so that

$$L_- = t^{\gamma-2} g_- .$$

We estimate for $0 \leq \sigma \leq \rho'$

$$\partial_t \|\omega^\sigma v'_-\|^2 = 2 \operatorname{Im} \left(\langle \omega^\sigma v'_-, \omega^\sigma L_2 v'_- \rangle + \langle \omega^\sigma v'_-, \omega^\sigma L_- v'_1 \rangle \right) . \quad (4.3)$$

By the estimates in the proof of Proposition 3.2 (see in particular (3.1) ; see also Remark 3.1), we obtain

$$\|\omega^\sigma v'_-(t)\| \leq E(t, a) \int_0^t dt' t'^{\gamma-2} \|\omega^\sigma g_- v'_1(t')\| . \quad (4.4)$$

We next estimate

$$\|\omega^\sigma g_- v'_1\| \leq C \|\omega^{n/2 \pm 0} g_-\| \|\omega^\sigma v'_1\| , \quad (4.5)$$

$$\|\omega^{n/2 \pm 0} g_{S-}\| \leq C t^{\rho-\gamma/2} \|\omega^\rho v_-\| \|\omega^\rho v_+\| . \quad (4.6)$$

In order to estimate g_{L-} , we use again Lemma 2.8, part (2). From the conservation law (2.23) and from the fact that $v_-(0) = 0$ we obtain (see (2.24))

$$|v(t)|_-^2 \equiv (1/2) (|v_2|^2 - |v_1|^2) = V_{1-}(t) + V_{2-}(t) \quad (4.7)$$

where (see (2.25) (2.26))

$$\begin{aligned} V_{1-}(t) &= - \int_0^t dt' \operatorname{Im} (\bar{v}_+ \Delta v_- + \bar{v}_- \Delta v_+) (t') , \\ V_{2-}(t) &= \int_0^t dt' \nabla \cdot s |v|_-^2 (t') . \end{aligned}$$

In the same way as in Section 2 (see (2.30)-(2.32)), we rewrite

$$V_{2-} = \nabla \cdot \int_0^t dt' (s - s_0) |v|_-^2 (t') + \nabla \cdot \int_0^t dt' s_0 |v|_-^2 (t') \quad (4.8)$$

and we substitute again (4.7) into the last term, so that

$$V_{2-} = V_{3-} + V_{4-} + V_{5-} \quad (4.9)$$

with

$$V_{3-} = \nabla \cdot \int_0^t dt' (s - s_0) |v|_-^2 (t') , \quad (4.10)$$

$$\begin{aligned} V_{4-} &= \nabla \cdot \int_0^t dt' s_0(t') V_{1-}(t') \\ &= -\nabla \cdot \int_0^t dt' \int_0^{t'} dt'' s_0(t') \operatorname{Im} (\bar{v}_+ \Delta v_- + \bar{v}_- \Delta v_+) (t'') , \end{aligned} \quad (4.11)$$

$$V_{5-} = \nabla \cdot \int_0^t dt' s_0(t') V_{2-}(t') = \nabla \cdot \int_0^t dt' \int_0^{t'} dt'' s_0(t') \nabla \cdot s |v|_-^2 (t'') . \quad (4.12)$$

By essentially the same estimates as in Lemma 2.8, part (2), we obtain from (4.5) (4.6)

$$\begin{aligned}
t^{\gamma-2} \|\omega^\sigma g_- v_1'\| &\leq C \left\{ a t^{\lambda_1-1} + a^3 t^{\lambda_0+\lambda_2-1} \right. \\
&\quad \left. + a^3 a_1^2 t^{2\lambda_0+\lambda_1-2} (1 + a^2 t^{\lambda_1}) \right\} a' \|v_-; L^\infty((0, t], H^\rho)\| \\
&\leq C a(1 + a^2)^2 (1 + a_1^2) a' t^{2\gamma+\lambda_1-2} \|v_-; L^\infty((0, t], H^\rho)\|
\end{aligned} \tag{4.13}$$

for $0 \leq \sigma \leq \rho'$, by keeping the dominant power of t in the last inequality (see (3.13)-(3.15)).

Substituting (4.13) into (4.4) yields (4.1).

□

We can now state the main result on the Cauchy problem at time zero for the equation (1.19).

Proposition 4.1. *Let $1/3 < \gamma < 1/2$ and $2 - 5\gamma/2 < \rho < n/2$, let $v_0 \in H^\rho$ and let s, s_0 and v_a be defined by (1.31)-(1.34) and (1.29) with $v_a(0) = v_0$. Then there exists $T > 0$ and there exists a unique solution $v \in \mathcal{C}([0, T], H^\rho)$ of the equation (1.19) with $L(v)$ defined by (1.30), with $v(0) = v_0$. One can ensure that*

$$\|v; L^\infty([0, T], H^\rho)\| \leq R = 2 \|v_0; H^\rho\| \tag{4.14}$$

$$C R^2 (1 + R^2)^3 T^{2\gamma+\lambda_1-1} = 1 \tag{4.15}$$

for some C independent of v_0 .

The proof is identical with that of Proposition 4.1 of I.

We finally translate the main result of Proposition 4.1 in terms of u_c and we derive additional bounds and regularity properties for u_c .

Proposition 4.2. *Let $1/3 < \gamma < 1/2$ and $2 - 5\gamma/2 < \rho < n/2$, let $v_0 \in H^\rho$. Let φ be defined by (1.28) with $\varphi(1) = 0$, with s_0 defined by (1.32) and v_a defined by (1.29) with $v_a(0) = 0$. Then there exists $T > 0$ and there exists a unique solution $u_c \in \mathcal{C}((0, T], H^\rho)$ of the equation (1.11) such that v defined by (1.12) satisfies the equation (1.19) with $L(v)$ defined by (1.30), with $v(0) = v_0$. Furthermore u_c satisfies the estimate*

$$\|u_c(t); H^\rho\| \leq C a_0 \left(1 + a_0^2 (1 + a_0^2 t^\gamma) t^{\gamma-1}\right)^{1+[\rho]} \tag{4.16}$$

for all $t \in (0, T]$, where $[\rho]$ is the integral part of ρ and

$$a_0 = \|v_0; H^\rho\| \quad .$$

Sketch of proof. The proof is the same as that of Proposition 4.2 of I, using the appropriate estimates of φ . As in the latter we estimate

$$\|\omega^\rho u_c\| \leq C \left(1 + \|\omega^{n/2} \varphi\|\right)^{1+[\rho]} \|\omega^\rho v\| \quad . \quad (4.17)$$

In the present case, φ can be written as $\varphi = \varphi_0 + \varphi_b + \varphi_c$ where the various terms are defined in analogy with (1.32)-(1.34) with ∇ omitted and are estimated in the same way as in Lemma 2.5, parts (1) and (3). One obtains

$$\|\omega^{n/2} \varphi\| \leq C \left(a_0^2 t^{\gamma-1} + a_0^2 a_1^2 t^{2\gamma-1}\right) \quad (4.18)$$

which together with (4.17) and with the fact that $a_1 \leq C a_0$ under the condition (4.15), implies (4.16). □

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